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Ambiguity in Lorentz Transformation and Reciprocal Symmetric Transformation as the Answer

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ABSTRACT

We have shown that successive Lorentz transformations (LT) lead to ambiguous values for time and space, because of Wigner rotation and associated non associativity of LT. We have proposed a reciprocal symmetric transformation (RST) which gives unique values for time and space. RST also gives a rotation comparable to Wigner rotation. RST is complex. We have shown that the imaginary part corresponds to spin of Dirac electron.

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INTRODUCTION

Ungar wrote (Ungar, 2006), "The *non-associativity* of Einstein's velocity addition is not widely known. A point in case is a recently published article (Sonego & Pin, 2005) in the journal in which its authors… wrongly assert that it is *associative*". Thomas rotation gives rise to "a *non associative group structure* for the set of relativistically admissible velocities" (Ungar, 1989). Wigner (1939) has exploited this non associativity, and the implied property (13.1) [see Appendix A below], to explain Thomas precession. Ahmad and Alam (2009) have studied this non associativity and its pathological implications. In this paper we intend to see how this non associativity leads to ambiguities in the values of space and time components when two non collinear Lorentz boosts are applied.

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We need an "Associative Lorentz Transformation" with an associative law of addition of velocities, which fulfills relativistic requirements (Lorentz invariance etc.), and should also have rotation properties similar to Wigner rotation, to accommodate Thomas precession (Moller, 1957). We shall see that Reciprocal Symmetric Transformation (Ahmad & Alam, 2007) (RST) fulfills our requirements and is also associative, which makes it free from pathologies of LT. RST is complex. We shall see that complex nature involves Pauli algebra and makes it conform to Dirac's electron theory and quantum mechanics. We have divided the paper into 4 parts. Part 1 (sections 2 to 7) deal with LT; Part 2 (section 8) deals with RST; Part 3 (sections 9 and 10) deals with Comparison between Rotations in LT and RST and Part 4 (section 11) deals with Compatibility between RST and Quantum Mechanics. Details of some calculations are given in appendices A and B (sections 13 and 14). In Appendix C (section 15) we have included a short note on reciprocal symmetry and the justification for calling the transformation reciprocal symmetric.

Part 1. Lorentz Transformation

LT IN ONE SPACE DIMENSION

We consider relativistic velocities which fulfill Einstein's condition

$$-c \le u \le c \tag{2.1}$$

For convenience we form functions

$$\frac{1+u/c}{\sqrt{1-(u/c)^2}} \tag{2.2}$$

Products of these functions give Lorentz-Einstein sum of velocities

$$\frac{1+u/c}{\sqrt{1-(u/c)^2}}\,\frac{1-v/c}{\sqrt{1-(v/c)^2}} = \frac{1+w/c}{\sqrt{1-(w/c)^2}} \tag{2.3}$$

Where w is the relativistic sum of velocities u and v

$$w/c = (u/c) \oplus (-v/c) = \frac{u/c - v/c}{1-u.v/c^2} \tag{2.4}$$

UPPER BOUND OF LENGTH AND LT

Definition 1: ct is the (expanding) upper bound length.
Definition 2: x is a Lorentz algebraic distance *x*, which obeys the condition

$$-ct \le x \le ct \tag{3.1}$$

(3.1) means we remain within the light cone.
Corresponding to (2.2) we form the functions

$$\frac{1+x/ct}{\sqrt{1-(x/ct)^2}} = \frac{ct+x}{\sqrt{(ct)^2 - x^2}} \tag{3.2}$$

Including a *t* in the numerator and the denominator we may write (2.2) in the form

$$\frac{1+v/c}{\sqrt{1-(v/c)^2}} = \frac{1+vt/(ct)}{\sqrt{1-(v/c)^2}} \qquad (3.3)$$

We exploit (3.3) to write the product

$$\begin{aligned}&\frac{1-v/c}{\sqrt{1-(v/c)^2}}\cdot\frac{ct+x}{\sqrt{(ct)^2-x^2}} = \\ &\frac{1}{\sqrt{(ct)^2-x^2}}\cdot\frac{c\{t-(x.v)/c^2\}+x-vt}{\sqrt{1-(v/c)^2}} = \frac{ct'+x'}{\sqrt{(ct)^2-x^2}}\end{aligned} \qquad (3.4)$$

Where t' and x' are the Lorentz transforms of t and x corresponding to (2.3)

$$t' = g\left(t - xv/c^2\right) \text{ and } x' = g(-vt + x) \qquad (3.5)$$

With

$$g = \frac{1}{\sqrt{1-(v/c)^2}}, \quad g' = \frac{1}{\sqrt{1-(u/c)^2}}, \quad g'' = \frac{1}{\sqrt{1-(w_L/c)^2}} \text{ and}$$

$$g\wedge = \frac{1}{\sqrt{(ct)^2-x^2}} \qquad (3.6)$$

We shall need the above definitions later.

(3.5) follows from (3.1). Therefore, LT (3.5) implies that we remain within the light cone.

General LT

So far we have considered motion in one space dimension only. For the general case corresponding to (2.3) and using (3.4) and introducing the notation for Lorentz rule of multiplication $\times_L$ (L for Lorentz) we have. Below in (4.2) we shall introducing the notation $+_L$ to represent Lorentz rule of addition

$$(g - g\mathbf{v}/c)\times_L (g' + g'\mathbf{u}/c) = (g'' + g''\mathbf{w}_L/c) \qquad (4.1)$$

Where $\mathbf{w}_L$ is given by

$$\mathbf{w}_L = (-\mathbf{v}) +_L \mathbf{u} = \frac{-\mathbf{v} + \mathbf{u}/g + \{1 - 1/g\}\dfrac{\mathbf{u.v}}{\mathbf{v}^2}\mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2} \qquad (4.2)$$

Corresponding to (3.4), we have for the general case

$$(g - g\mathbf{v}/c)\times_L (ct + \mathbf{x}) = ct' + \mathbf{x}' \qquad (4.3)$$

t' and $\mathbf{x}'$ will fulfill our relativistic requirements (Rindler, 1966) if

$$t' = g\left(t - \mathbf{x}\cdot\mathbf{v}/c^2\right) \text{ and } \mathbf{x}' = -g\mathbf{v}t + g\frac{\mathbf{x}\cdot\mathbf{v}}{v^2}\mathbf{v} + \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{v}}{v^2}\mathbf{v} \qquad (4.4)$$

(4.3) and (4.4) agree with (2.4) and (3.5) in case of collinear motion.

General LT in Matrix Form

We may write (4.2) and (4.4) as a matrix product

$$\begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} g \\ -g\mathbf{v}/c \end{pmatrix} \times_L \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} g & -g\mathbf{v}^{\mathrm{T}}/c \\ -g\mathbf{v}/c & 1+\{g-1\}\dfrac{\mathbf{v}}{v^2}\mathbf{v}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} \qquad (5.1)$$

Where $\mathbf{v}^{\mathrm{T}}$ is the transpose of $\mathbf{v}$ and (5.1) defines the operation represented by $\times_L$ so that

$$\begin{pmatrix} g \\ -g\mathbf{v}/c \end{pmatrix} \times_L = \begin{pmatrix} g & -g\mathbf{v}^{\mathrm{T}}/c \\ -g\mathbf{v}/c & 1+\{g-1\}\dfrac{\mathbf{v}}{v^2}\mathbf{v}^{\mathrm{T}} \end{pmatrix} \qquad (5.2)$$

(5.1) defines the operation represented by $\times_L$. The column on the left of $\times_L$ has to be replaced by the square matrix (Kyrala, 1967) in (5.2). Then, it is a matrix multiplication. Corresponding to (4.1) and (4.2) we have

$$\begin{pmatrix} g'' \\ g''\mathbf{w}_L/c \end{pmatrix} = \begin{pmatrix} g \\ -g\mathbf{v}/c \end{pmatrix} \times_L \begin{pmatrix} g' \\ g'\mathbf{u}/c \end{pmatrix} = \begin{pmatrix} g & -g\mathbf{v}^{\mathrm{T}}/c \\ -g\mathbf{v}/c & 1+\{g-1\}\dfrac{\mathbf{v}}{v^2}\mathbf{v}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} g' \\ g'\mathbf{u}/c \end{pmatrix}$$

(5.3)

Where

$$g'' = g'.g\left(1-\mathbf{u}\cdot\mathbf{v}/c^2\right) = \frac{1}{\sqrt{1-(w_L/c)^2}} \qquad (5.4)$$

Successive LTs

Consider the successive transforms

$$(f'-f'\mathbf{y}/c)\times_L \{(g-g\mathbf{v}/c)\times_L (ct+\mathbf{x})\} = (f'-f'\mathbf{y}/c)\times_L (ct'+\mathbf{x}') = (ct''+\mathbf{x}'')$$

(6.1)

And

$$\{(f'-f'\mathbf{y}/c)\times_L (g-g\mathbf{v}/c)\}\times_L (ct+\mathbf{x}) = (f\wedge - f\wedge \mathbf{m}_L/c)\times_L (ct+\mathbf{x}) = (ct\wedge + \mathbf{x}\wedge)$$

(6.2)

Where using the notation of (4.2)

$$\mathbf{m}_L = \mathbf{y} +_L \mathbf{v} = \frac{\mathbf{y}+\mathbf{u}/f' + \{1-1/f'\}\dfrac{\mathbf{v.y}}{\mathbf{y}^2}\mathbf{y}}{1+\mathbf{y}\cdot\mathbf{v}/c^2} \qquad (6.3)$$

and

$$f' = \frac{1}{\sqrt{1-(y/c)^2}}, \quad f\wedge = \frac{1}{\sqrt{1-(m_L/c)^2}} \qquad (6.4)$$

Left hand sides of (6.1) and (6.2) differ only in the way they are arranged. To write the middle part of (6.1) we have used (5.1) and for the middle part of (6.2) we have used the algebra of (5.3) with (6.3). t'' and $\mathbf{x}''$ are given by

$$t'' = f'\left(t' - \mathbf{x}'\cdot\mathbf{y}/c^2\right) \text{ and } \mathbf{x}'' = -f'\mathbf{y}t' + f'\frac{\mathbf{x}'\cdot\mathbf{y}}{y^2}\mathbf{y} + \mathbf{x}' - \frac{\mathbf{x}'\cdot\mathbf{y}}{y^2}\mathbf{y} \tag{6.5}$$

Using (4.4) and

$$f\wedge = f'g\left(1 + \mathbf{v}\cdot\mathbf{y}/c^2\right) \tag{6.6}$$

Working out the algebra we have from (6.1)

$$t'' = f'g\left(1 + \frac{\mathbf{v.y}}{c^2}\right)t - f'g\left(\frac{(\mathbf{x.v}).(\mathbf{v.y})}{v^2c^2} + \frac{\mathbf{x.v}}{c^2}\right) + f'\left(\frac{\mathbf{x.y}}{c^2} - \frac{(\mathbf{x.v}).(\mathbf{v.y})}{v^2c^2}\right) \tag{6.7}$$

$$\mathbf{x}'' = -f'g\left\{\mathbf{y} + \frac{\mathbf{v.y}}{y^2}\mathbf{y}\right\}t + f'g\left\{\frac{(\mathbf{x.v}).(\mathbf{v.y})}{v^2y^2} + \left(\frac{\mathbf{x.v}}{c^2}\right)\right\}\mathbf{y} + f'\left\{\frac{\mathbf{x.y}}{y^2} - \frac{(\mathbf{x.v}).(\mathbf{v.y})}{v^2y^2}\right\}\mathbf{y}$$

$$-g\left\{\mathbf{v} - \frac{\mathbf{v.y}}{y^2}\mathbf{y}\right\}t$$

$$+g\left\{\left(\frac{\mathbf{x.v}}{v^2}\right)\mathbf{v} - \frac{(\mathbf{x.v}).(\mathbf{v.y})}{v^2y^2}\mathbf{y}\right\} + \left(\mathbf{x} - \frac{\mathbf{x.v}}{v^2}\mathbf{v}\right) + \left\{\frac{(\mathbf{x.v}).(\mathbf{v.y})}{v^2y^2} - \frac{\mathbf{x.y}}{y^2}\right\}\mathbf{y} \tag{6.8}$$

(6.2) gives

$$t\wedge = \frac{t - \mathbf{x}\cdot\mathbf{m}_L/c^2}{\sqrt{1-(m_L/c)^2}} \tag{6.9}$$

$$\mathbf{x}\wedge = \frac{-\mathbf{m}_L t + \dfrac{\mathbf{x}\cdot\mathbf{m}_L}{\mathbf{m}_L^2}\mathbf{m}_L}{\sqrt{1-(m_L/c)^2}} + \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{m}_L}{m_L^{\ 2}}\mathbf{m}_L \tag{6.10}$$

Lorentz invariance condition is fulfilled by both the sets:

$$(ct'')^2 - (\mathbf{x}'')^2 = (ct\wedge)^2 - (\mathbf{x}\wedge)^2 = (ct)^2 - \mathbf{x}^2 \tag{6.11}$$

NON-ASSOCIATIVITY AND AMBIGUITY IN LT

Comparison between (6.7) and (6.9) and between (6.8) and (6.10) show (Ahmad & Alam, 2009)

$$t'' \neq t\wedge \text{ and } \mathbf{x}'' \neq \mathbf{x}\wedge \tag{7.1}$$

(6.1), (6.2) and the inequality (7.1) show that LT is not associative and that it leads to ambiguities. We need an associative transformation which will be free from these ambiguities.

When **x**, **v** and **y** are collinear (Moller, 1957)

$$t'' = t\wedge \text{ and } \mathbf{x}'' = \mathbf{x}\wedge \tag{7.2}$$

Part 2. Reciprocal Symmetric Transformation

Associative Transformation

Postulate: We replace $\times_L$ of (4.1) by $\times_{RS}$ defined (Ahmad & Alam, 2009) below. [See (14.6) of Appendix B]

$$(a+\mathbf{u})\times_{RS}(b+\mathbf{v}) = (ab+\mathbf{u}\cdot\mathbf{v}) + (b\mathbf{u}+a\mathbf{v}+i\mathbf{u}\times\mathbf{v}) \qquad (8.1)$$

We shall call $\times_{RS}$ the reciprocal symmetric multiplication and the corresponding transformation RST. In Appendix C, we shall give the justification for calling it reciprocal symmetric.

We now replace (6.1) and (6.2) by

$$(f'-f'\mathbf{y}/c)\times_{RS}\{(g-g\mathbf{v}/c)\times_{RS}(ct+\mathbf{x})\} = (f'-f'\mathbf{y}/c)\times_{RS}(ct^*+\mathbf{x}^*) = (ct^{**}+\mathbf{x}^{**}) \qquad (8.2)$$

And

$$\{(f'-f'\mathbf{y}/c)\times_{RS}(g-g\mathbf{v}/c)\}\times_{RS}(ct+\mathbf{x}) = (f\wedge - f\wedge\mathbf{m}_{RS}/c)\times_{RS}(ct+\mathbf{x}) = (ct^{**}+\mathbf{x}^{**}) \qquad (8.3)$$

The right hand sides are equal because, unlike $\times_L$, $\times_{RS}$ is associative.

We use (8.1) repeatedly to determine t^{**} and $\mathbf{x}^{**}$ given by (8.2) and (8.3) [See Appendix B]. Lorentz invariance requirement (6.11) is fulfilled.

$$(ct^{**})^2-(\mathbf{x}^{**})^2 = (ct)^2-\mathbf{x}^2 \qquad (8.4)$$

Part 3. Comparison between Rotations in LT and RST

Rotation in LT

In this section we want to study the nature and the quantity of the rotation which takes place when a Lorentz boost is given. For Lorentz 4-vector we write $R_L = ct+\mathbf{x}$ and for the transformed vector corresponding to (4.4) we write

$$R'_L = ct'+\mathbf{x}' = g(ct-\mathbf{x}\cdot\mathbf{v}/c) - g\mathbf{v}t + g\frac{\mathbf{x}\cdot\mathbf{v}}{v^2}\mathbf{v} + \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{v}}{v^2}\mathbf{v} \qquad (9.1)$$

In the limit $c\to\infty$ we go over to Galilean transformation and there is no rotation. To see the maximum rotation we go to the limit $c\to 0$. We want to study the mathematical nature of it, so the physical interpretation (if any) of this limit does not interest us. We are interested in the rotation part only so we choose the dimension less quantity $-\dfrac{cR'_L}{g|x|.|v|}$

Using the limit $1/g \xrightarrow[c\to 0]{} ic/|\mathbf{v}|$ we have

$$-\frac{cR'_L}{g|x|.|v|} \xrightarrow[c\to 0]{} \frac{\mathbf{x}\cdot\mathbf{v}}{|\mathbf{x}|.|\mathbf{v}|} - i\frac{\mathbf{x}-\dfrac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}^2}\mathbf{v}}{|\mathbf{x}|} = Cos\theta - \mathbf{ip}Sin\theta \qquad (9.2)$$

Where θ is the angle between $\mathbf{x}$ and $\mathbf{v}$. $\mathbf{p}$ gives the direction of the axis of rotation. It is a unit vector orthogonal to $\mathbf{v}$ in the plane of $\mathbf{x}$ and $\mathbf{v}$.

ROTATION IN RST

To study the rotation corresponding to (9.2) for the reciprocal symmetric case, we replace R'_L of (9.1) by $R*_{RS} = ct*+\mathbf{x}*$ where $ct*$ and $\mathbf{x}*$ are given by (14.10).

$$R*_{RS} = ct*+\mathbf{x}* = g(ct - \mathbf{x}\cdot\mathbf{v}/c) + g\{\mathbf{x} - \mathbf{v}t - \mathbf{i}\mathbf{x}\times\mathbf{v}/c\} \qquad (10.1)$$

Corresponding to (9.2) we now have (Ahmad, *etl.* 2007)

$$-\frac{cR*_{RS}}{g\,|\mathbf{x}|.|\mathbf{v}|} \xrightarrow[c\to 0]{} \frac{\mathbf{x}\cdot\mathbf{v}}{|\mathbf{x}|.|\mathbf{v}|} + i\frac{\mathbf{x}\times\mathbf{v}}{|\mathbf{x}|.|\mathbf{v}|} = Cos\theta - i\mathbf{q}Sin\theta \qquad (10.2)$$

Resemblance between (9.2) and (10.2) is close. $\mathbf{q}$ gives the direction of the axis of rotation. It is a unit vector orthogonal to the plane of $\mathbf{x}$ and $\mathbf{v}$.

Part 4. Compatibility between RST and Quantum Mechanics

COMPATIBILITY BETWEEN RST AND DIRAC THEORY

We write Dirac's relativistic equation as (Schiff, 1970)

$$\left(E - c\boldsymbol{\alpha}\cdot\mathbf{p} - \beta mc^2\right)\psi = 0 \qquad (11.1)$$

For the purpose of comparison with (14.1) we set $m = 0$ and we write (11.1) as

$$\left(E\sigma_0 - c\boldsymbol{\sigma}\cdot\mathbf{p}\right)\psi = 0 \qquad (11.2)$$

We operate on the left by $(E + c\boldsymbol{\sigma}\cdot\mathbf{p})$ to get

$$\left(E\sigma_0 + c\boldsymbol{\sigma}\cdot\mathbf{p}\right)\left(E\sigma_0 - c\boldsymbol{\sigma}\cdot\mathbf{p}\right)\psi = \left(E^2 - (c\mathbf{p})^2\right)\psi \qquad (11.3)$$

(11.3) will be correct if σ s of (11.2) have the properties of (14.3) and (14.4). To see the correspondence to spin consider the product of type

$$\left(b\sigma_0 + \boldsymbol{\sigma}\cdot\mathbf{B}\right)\times_{RS}\left(c\sigma_0 + \boldsymbol{\sigma}\cdot\mathbf{C}\right) = bc + \mathbf{B}\cdot\mathbf{C} + \boldsymbol{\sigma}\cdot\left(b\mathbf{C} + c\mathbf{B} + i\mathbf{B}\times\mathbf{C}\right) \qquad (11.4)$$

In the presence of electromagnetic field, in Dirac theory, we come across terms like (Schiff, 1970)

$$\left(\boldsymbol{\sigma}\cdot\mathbf{B}\right)\times_{RS}\left(\boldsymbol{\sigma}\cdot\mathbf{C}\right) = \mathbf{B}\cdot\mathbf{C} + i\boldsymbol{\sigma}\cdot\left(\mathbf{B}\times\mathbf{C}\right) \qquad (11.5)$$

The last term gives spin (Ahmad 2007). Therefore, the imaginary cross term of (8.1) corresponds to spin (Ahmad, *etl.* 2007). We conclude that RST is closer to quantum mechanics than LT.

CONCLUSION

Inequalities (7.1) show that LT is non associative and that it leads to ambiguities. RST is associative (section 8). RST also gives a rotation similar to Wigner rotation, and the extreme case of $c \to 0$, the two rotations are quantitatively the same, but with different axes of rotation. RST and Dirac's electron theory share the same algebraic properties. Therefore, RST promises to bring relativity and quantum mechanics closer (Ahmad 2006).

Part 5. Appendices

Appendix A: Mathematical Property of Associative Addition

Theorem:

If $+_{RS}$ stands for an associative addition

Then

$$-(\mathbf{y}+_{RS}\mathbf{v})=(-\mathbf{v})+_{RS}(-\mathbf{y}) \tag{13.1}$$

Proof:

Using associativity of $+_{RS}$ and $\mathbf{v}+_{RS}(-\mathbf{v})=0$, we have

$$(\mathbf{y}+_{RS}\mathbf{v})+_{RS}\{(-\mathbf{v})+_{RS}(-\mathbf{y})\}=\mathbf{y}+_{RS}\{\mathbf{v}+_{RS}(-\mathbf{v})\}+_{RS}(-\mathbf{y})=\mathbf{y}+_{RS}(-\mathbf{y})=0 \tag{13.2}$$

Therefore,

$$(\mathbf{y}+_{RS}\mathbf{v})+_{RS}\{(-\mathbf{v})+_{RS}(-\mathbf{y})\}=0 \tag{13.3}$$

Also we have

$$\mathbf{y}+_{RS}\mathbf{v}+_{RS}\{-(\mathbf{y}+_{RS}\mathbf{v})\}=0 \tag{13.4}$$

Comparison between (13.4) and (13.5) shows

$$-(\mathbf{y}+_{RS}\mathbf{v})=(-\mathbf{v})+_{RS}(-\mathbf{y}) \tag{13.5}$$

Corollary 1: If addition $+_L$ is not commutative and if $-(\mathbf{y}+_L\mathbf{v})=(-\mathbf{y})+_L(-\mathbf{v})$, then addition $+_L$ is not associative.

Proof:

We assume that $+_L$ is associative. Then, by the theorem above

$$-(\mathbf{y}+_L\mathbf{v})=(-\mathbf{v})+_L(-\mathbf{y}) \tag{13.6}$$

But

$$-(\mathbf{y}+_L\mathbf{v})=(-\mathbf{y})+_L(-\mathbf{v})\neq(-\mathbf{v})+_L(-\mathbf{y}) \tag{13.7}$$

Therefore,(13.7) contradicts (13.6). Therefore, the assumption that $+_L$ is associative is wrong. The proposition that $+_L$ is associative is not correct.

Corollary 2: LT $+_L$ as defined below is not associative.

$$\mathbf{m}_L=\mathbf{y}+_L\mathbf{v}=\frac{\mathbf{y}+\mathbf{v}/g'+\{1-1/g'\}\frac{\mathbf{v}\cdot\mathbf{y}}{\mathbf{y}^2}\mathbf{y}}{1-y\cdot\mathbf{v}/c^2} \tag{13.8}$$

Proof:

By (10.8)

$$-(\mathbf{y}+_L\mathbf{v})=(-\mathbf{y})+_L(-v) \tag{13.9}$$

Therefore, by Corollary 1, LT $+_L$ as defined by (13.8) is not associative.

Wigner (1939) has exploited inequality (13.7) of LT to explain Thomas precession.

APPENDIX B: PAULI QUATERNION 4-VECTORS AND RST

Postulate: We postulate (Ahmad & Alam, 2009) that the 0+3 [scalar+Cartesian] vectors $(a+\mathbf{u})$ etc. of (8.1) are, in fact, Pauli Quaternion (Rastall, 1964) 4-vectors.

$$(a+\mathbf{u}) \Rightarrow (a\sigma_0 + \mathbf{u}\cdot\boldsymbol{\sigma}) = a\sigma_0 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z \tag{14.1}$$

where

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{14.2}$$

σ s have the following properties

$$-\sigma_y\sigma_x = \sigma_x\sigma_y = i\sigma_z \text{ with cyclic permutations} \tag{14.3}$$

$$\sigma_0\sigma_x = \sigma_x\sigma_0 = \sigma_x \text{ and } \sigma_x\sigma_x = 1 \text{ and also for } y \text{ and } z \tag{14.4}$$

We replace $\times_L$ of (4.1) by $\times_{RS}$ (*RS* for Reciprocal Symmetric), and using (14.1) through (14.3) we get

$$(a+\mathbf{u})\times_{RS}(b+\mathbf{v}) = (ab+\mathbf{u}\cdot\mathbf{v})\sigma_0 + (b\mathbf{u}+a\mathbf{v}+i\mathbf{u}\times\mathbf{v})\cdot\boldsymbol{\sigma} \tag{14.5}$$

We shall also use $\times_{RS}$ to mean (14.3) without the σ s. i.e.

$$(a+\mathbf{u})\times_{RS}(b+\mathbf{v}) = (ab+\mathbf{u}\cdot\mathbf{v}) + (b\mathbf{u}+a\mathbf{v}+i\mathbf{u}\times\mathbf{v}) \tag{14.6}$$

We replace (6.1) and (6.2) by

$$(f'-f'\mathbf{y}/c)\times_{RS}\{(g-g\mathbf{v}/c)\times_{RS}(ct+\mathbf{x})\} = (f'-f'\mathbf{y}/c)\times_{RS}(ct^*+\mathbf{x}^*) = (ct^{**}+\mathbf{x}^{**}) \tag{14.7}$$

And

$$\{(f'-f'\mathbf{y}/c)\times_{RS}(g-g\mathbf{v}/c)\}\times_{RS}(ct+\mathbf{x}) = (f\hat{}-f\hat{}\,\mathbf{m}_{RS}/c)\times_{RS}(ct+\mathbf{x}) = (ct^{**}+\mathbf{x}^{**}) \tag{14.8}$$

The right hand sides are equal because, unlike $\times_L$, $\times_{RS}$ is associative.

In place **m** of (6.4) given by (4.3) we now have, using (14.1)--(14.3), $\mathbf{m}_{RS}$ below

$$\mathbf{m}_{RS} = \mathbf{y} +_{RS} \mathbf{v} = \frac{\mathbf{y}+\mathbf{v}+i\mathbf{y}\times\mathbf{v}/c}{1+\mathbf{y}\cdot\mathbf{v}/c^2} \tag{14.9}$$

In place of t' and $\mathbf{x}'$ we now have, for t^* and $\mathbf{x}^*$ of (14.7) given by

$$t^* = t' = \frac{t-\mathbf{x}\cdot\mathbf{v}/c^2}{\sqrt{1-(v/c)^2}} \quad\text{and}\quad \mathbf{x}^* = \frac{\mathbf{x}-\mathbf{v}t-i\mathbf{x}\times\mathbf{v}/c}{\sqrt{1-(v/c)^2}} \tag{14.10}$$

Working out the algebra we have for t^{**} and $\mathbf{x}^{**}$ given by (14.7) and (14.8) we have

$$t^{**} = \frac{t^*-\mathbf{x}^*\cdot y/c^2}{\sqrt{1-(y/c)^2}} = \frac{t\{1+(v/c^2)\}-\mathbf{x}\cdot\mathbf{v}/c^2-\mathbf{x}\cdot\mathbf{y}/c^2+i(\mathbf{x}\times\mathbf{v})\cdot\mathbf{y}/c^3}{\sqrt{1-(v/c)^2}\sqrt{1-(y/c)^2}} \tag{14.11}$$

And

$$\mathbf{x}^{**} = \frac{\mathbf{x} - \mathbf{v}t - \mathbf{y}t + (\mathbf{x}\cdot\mathbf{v}/c^2)\mathbf{y} + \{(\mathbf{x}\times\mathbf{v})\times\mathbf{y}/c^2\} - i\{\mathbf{x}\times\mathbf{v}/c + \mathbf{x}\times\mathbf{y}/c - (\mathbf{v}\times\mathbf{y})t/c\}}{\sqrt{1-(v/c)^2}\sqrt{1-(y/c)^2}} \tag{14.12}$$

We again have corresponding to (6.11)

$$(ct^{**})^2 - (\mathbf{x}^{**})^2 = (ct)^2 - \mathbf{x}^2 \tag{14.13}$$

APPENDIX C: RECIPROCAL SYMMETRY

Consider the relative velocity (2.4) in one dimension. We shall set $c = 1$ in this section

$$w = (-v)\oplus u = \frac{-v+u}{1-u.v} \tag{15.1}$$

We note that

$$w = (-1/v)\oplus(1/u) = \frac{-1/v+1/u}{1-1/(u.v)} = \frac{-v+u}{1-u.v} = (-v)\oplus u \tag{15.2}$$

When v and u are replaced by their reciprocals, the sum remains unchanged. This property we call reciprocal symmetry. Consider the reciprocal symmetric sum corresponding to (4.2), using relation (14.9)

$$\mathbf{w}_{RS} = -\mathbf{v} +_{RS} \mathbf{u} = \frac{-\mathbf{v}+\mathbf{u}-i\mathbf{v}\times\mathbf{u}}{1-\mathbf{v}\cdot\mathbf{u}} \tag{15.3}$$

We shall call reciprocal of $\mathbf{v}$ any quantity $\mathbf{v}'$ which fulfills the condition

$$\mathbf{v}\cdot\mathbf{v}' = 1 \tag{15.4}$$

We choose the reciprocal

$$\mathbf{v}' = \frac{\mathbf{g}+i\mathbf{g}\times\mathbf{v}}{\mathbf{g}\cdot\mathbf{v}} \tag{15.5}$$

Theorem:

$$-\mathbf{v}' +_{RS} \mathbf{u}' = -\mathbf{v} +_{RS} \mathbf{u} \tag{15.6}$$

Proof: Let

$$\mathbf{v}^\wedge = \frac{\mathbf{g}+\mathbf{v}+i\mathbf{g}\times\mathbf{v}}{1+\mathbf{g}\cdot\mathbf{v}} = \mathbf{g} +_{RS} \mathbf{v} \tag{15.7}$$

We observe that

$$Lt_{g\to\infty}\mathbf{v}^\wedge = \mathbf{v}' \quad \text{or} \quad \mathbf{g} +_{RS} \mathbf{v} \xrightarrow[g\to\infty]{} \mathbf{v}' \tag{15.8}$$

Therefore, instead of $-\mathbf{v}' +_{RS} \mathbf{u}'$ we write $Lt_{g\to\infty} -\mathbf{v}^\wedge +_{RS} \mathbf{u}^\wedge$ and we write for the left hand side of (15.6)

$$Lt_{g\to\infty}\{-(\mathbf{g} +_{RS} \mathbf{v}) +_{RS} (\mathbf{g} +_{RS} \mathbf{u})\} \tag{15.9}$$

Using (13.5) and associativity we get

$$Lt_{g\to\infty}\{[(-\mathbf{v})+_{RS}(-\mathbf{g})]+_{RS}[\mathbf{g}+_{RS}\mathbf{u}]\}=Lt_{g\to\infty}\{-\mathbf{v}+_{RS}\mathbf{u}\}=-\mathbf{v}+_{RS}\mathbf{u} \qquad (15.10)$$

Using (15.8) and (15.10) we get (15.6).

One may also write (15.6) using (15.5). This gives

$$\mathbf{w}_{RS}=-\mathbf{v}'+_{RS}\mathbf{u}'=\frac{-(\mathbf{g}\cdot\mathbf{u}).(\mathbf{g}+i\mathbf{g}\times\mathbf{v})+(\mathbf{g}\cdot\mathbf{v}).(\mathbf{g}+i\mathbf{g}\times\mathbf{u})-i(\mathbf{g}+i\mathbf{g}\times\mathbf{v})\times(\mathbf{g}+i\mathbf{g}\times\mathbf{u})}{(\mathbf{g}\cdot\mathbf{u}).(\mathbf{g}\cdot\mathbf{v})-(\mathbf{g}+i\mathbf{g}\times\mathbf{v})\cdot(\mathbf{g}+i\mathbf{g}\times\mathbf{u})} \qquad (15.11)$$

Working out the algebra we find (15.6).

Relation (15.6) justifies our calling the sum defined by (15.3) reciprocal symmetric.

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