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Rciprocal Property of Different Types of Lorentz Transformations

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ABSTRACT

Lorentz transformation is the basic tool for the study of Relativistic mechanics. There are different types of Lorentz transformations such as Special, Most general, Mixed number, Geometric product, and Quaternion Lorentz transformations. In this paper we have studied the reciprocal property of the above Lorentz transformations where the velocity of a particle is very close to the speed of light.

Key Words: Lorentz transformation and Reciprocal Property.

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1 INTRODUCTION

1.1 Special Lorentz transformation

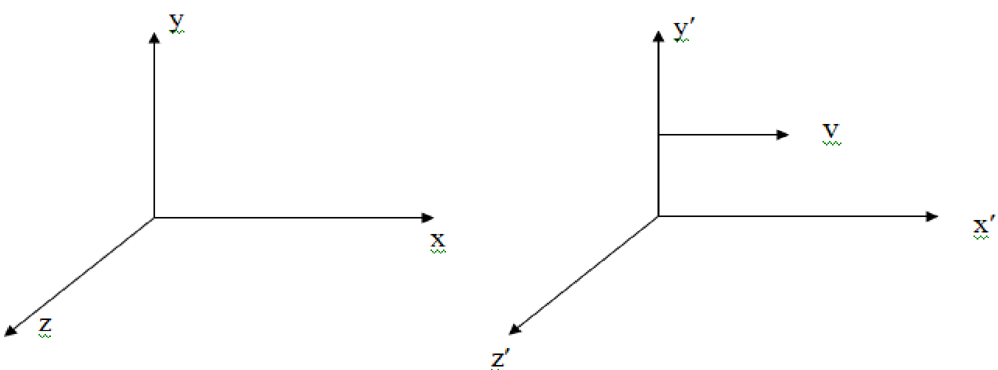

Figure 1: The frame S is at rest and the frame S′ is moving with respect to S with uniform velocity V along x-axis.

Consider two inertial frames of Refeence S and S' where the frame S is at rest and the frame S' is moving along X-axis with velocity V with respect to S frame. The space and time coordinates of S and S' are (x, y, z, t) and (x′, y′, z′, t′) respectively.Then the relation between the coordinates of S and S' is called special Lorentz Transformation which can be written as [1]

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \; y' = y, \quad z' = z \tag{1}$$

$$t' = \frac{t - \frac{V x}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

and

$$x = \frac{x' + V\,t'}{\sqrt{1 - \frac{V^2}{c^2}}}, \; y = y', \; z = z', \tag{2}$$

$$t = \frac{t' + \frac{V x'}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

1.2 Most General Lorentz Tranformation

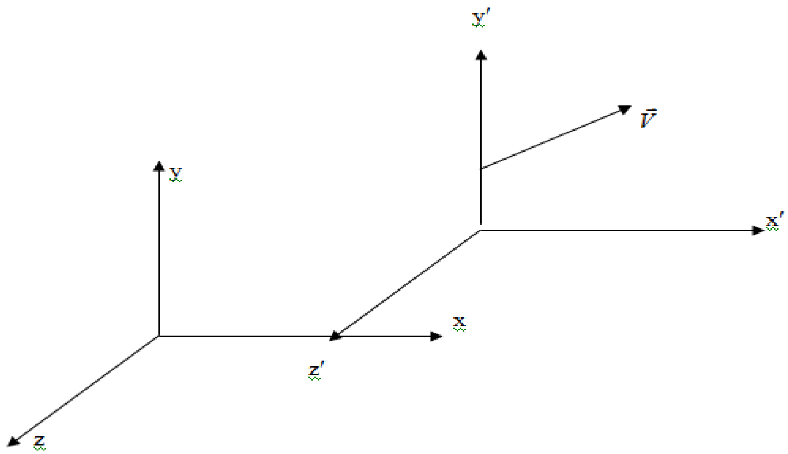

Figure -2: The frame S is at rest and the frame S′ is moving with respect to S with uniform velocity $\vec{V}$ along any arbitrary direction.

When the velocity $\vec{V}$ of S' with respect to the S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Then the relation between the coordinates of S and S' is called most general Lorentz Transformation which can be written as [2]

$$\vec{X}' = \vec{X} + \vec{V}[\{\frac{(\vec{X}.\vec{V})}{V^2}\}\{(1-\frac{V^2}{c^2})^{-1/2} - 1\} - t(1-\frac{V^2}{c^2})^{-1/2}]$$
$$t' = (1-\frac{V^2}{c^2})^{-1/2}\{t - \frac{(\vec{V}.\vec{X})}{c^2}\} \tag{3}$$

and

$$\vec{X} = \vec{X}' + \vec{V}'[\{\frac{(\vec{X}'.\vec{V}')}{V'^2}\}\{(1-\frac{V^2}{c^2})^{-1/2} - 1\} - t'(1-\frac{V^2}{c^2})^{-1/2}]$$
$$t = (1-\frac{V^2}{c^2})^{-1/2}\{t' - \frac{(\vec{V}'.\vec{X}')}{c^2}\} \tag{4}$$

Where $\vec{V}' = -\vec{V}$

1.3 Mixed Number Lorentz Transformation

Mixed number [3-7] α is the sum of a scalar x and a vector $\vec{A}$.

i.e. $\alpha = x + \vec{A}$

The product of two mixed numbers is defined as [3-7]

$$\alpha\beta = (x+\vec{A})(y+\vec{B}) = xy + \vec{A}.\vec{B} + x\vec{B} + y\vec{A} + i\vec{A}\times\vec{B} \tag{5}$$

Taking $x = y = 0$, we get from equation (5)

$$\vec{A}\otimes\vec{B} = \vec{A}.\vec{B} + i\vec{A}\times\vec{B} \tag{6}$$

This product is called mixed product [7] and the symbol $\otimes$ is chosen for it. We can generate a type of most general Lorentz transformation using this mixed product.

Using $c = 1$ and $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

in equations(1) and (2), we can write

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx) \end{aligned} \tag{7}$$

$$\begin{aligned} x &= \gamma(x' + vt') \\ t &= \gamma(t' + vx') \end{aligned} \tag{8}$$

Now from equation (7), we get

$$t' + x' = \gamma(t - vx + x - vt)$$

$$or,\ t' + x' = \gamma[(t + x) - v(t + x)] \tag{9}$$

Using (t' + x') = p' and (t + x) = p in equation (9), we can write

$$p' = \gamma(p - pv) \tag{10}$$

Now from equation (8), we get

$$t + x = \gamma(t' + vx' + x' + vt')$$

$$or,\ t + x = \gamma[(t' + x') + v(t' + x')] \tag{11}$$

Using $(t' + x') = p'$ and $(t + x) = p$ in equation (11), we can write

$$p = \gamma(p' + p'v) \tag{12}$$

In the case of the most general Lorentz transformation, the velocity $\vec{V}$ of S' with respect to S is not along the X-axis; i.e., the velocity $\vec{V}$ has three components, V_x, V_y, and V_z. Let in this case Z and Z' be the space parts in the S and S' frames, respectively. In this case equation (10) can be written as

$$P' = \gamma(P - P\vec{V}) \tag{13}$$

Where, $P' = (t' + \vec{Z}')$ and $P = (t + \vec{Z})$ are two mixed numbers [5].

Therefore,

$$(t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (t + \vec{Z})\vec{V}]$$

$$or,\ (t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (t + \vec{Z})(0 + \vec{V}) \tag{14}$$

Using equation (6), we can write

$$(t + \vec{Z})(0 + \vec{V}) = \vec{Z}.\vec{V} + t\vec{V} + i\vec{Z} \times \vec{V} \tag{15}$$

From (14) and (15) we get,

$$(t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (\vec{Z}.\vec{V} + t\vec{V} + i\vec{Z}\,.\,\vec{V})]$$

$$or,\ (t' + \vec{Z}') = \gamma(t - \vec{Z}\,.\,\vec{V}) + \gamma(\vec{Z} - t\vec{V} - i\vec{Z} \times \vec{V}) \tag{16}$$

Equating the scalar and vector part from the both sides of equation (16), we can write

$$\begin{aligned} t' &= \gamma(t - \vec{Z}.\vec{V}) \\ \vec{Z}' &= \gamma(\vec{Z} - t\vec{V} - i\vec{Z} \times \vec{V}) \end{aligned} \tag{17}$$

Similarly, we can show that

$$\begin{aligned} t &= \gamma(t' + \vec{Z}'.\,\vec{V}) \\ \vec{Z} &= \gamma(\vec{Z}' + t'\vec{V} + i\vec{Z}' \times \vec{V}) \end{aligned} \tag{18}$$

Equations (17) and (18) are the mixed-number Lorentz transformation.

1.4 Geometric Product Lorentz Transformation

Bidyut Kumar Datta and his co-workers defined the geometric product of vectors as [8, 9]

$$\vec{A}\vec{B} = \vec{A}.\vec{B} + \vec{A} \wedge \vec{B}$$

where $\vec{A}$ and $\vec{B}$ are two vectors. We use the symbol $\times$ instead of the symbol $\wedge$. Therefore, the geometric product of vectors can be written as

$$\vec{A}\vec{B} = \vec{A}.\vec{B} + \vec{A}\times\vec{B} \tag{19}$$

We can also generate a type of most general Lorentz transformation using this geometric product.

In this case the velocity $\vec{V}$ of S' with respect to S also has three components, V_x, V_y, and V_z as the most general Lorentz transformation. Let in this case $\vec{Z}$ and $\vec{Z}'$ be the space parts in the S and S' frames respectively. Equation (14) is

$$(t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (t + \vec{Z})(0 + \vec{V})]$$

$$or,\ (t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (t\vec{V}) - \vec{Z}\vec{V}] \tag{20}$$

From equation (19) the geometric product of two vectors is

$$\vec{A}\vec{B} = \vec{A}.\vec{B} + \vec{A}\times\vec{B}$$

Therefore, we can write

$$(\vec{Z}\vec{V}) = \vec{Z}.\vec{V} + \vec{Z}\times\vec{V} \tag{21}$$

From (20) and (21) we get,

$$(t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (\vec{Z}.\vec{V} + t\vec{V} + \vec{Z}\times\vec{V})]$$

$$or,\ (t' + \vec{Z}') = \gamma(t - \vec{Z}.\vec{V}) + \gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}) \tag{22}$$

Equating the scalar and vector part from the both sides of equation (22), we can write

$$t' = \gamma(t - \vec{Z}.\vec{V})$$
$$\vec{Z}' = \gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V}) \tag{23}$$

Similarly, we can show that

$$t = \gamma(t' + \vec{Z}'.\vec{V})$$
$$\vec{Z} = \gamma(\vec{Z}' + t'\vec{V} + \vec{Z}' \times \vec{V}) \tag{24}$$

Equations (23) and (24) are the geometric product Lorentz transformation.

1.5 Quaternion Lorentz Transformation

The quaternion can also be written as the sum of a scalar and a vector [10]

i.e., $\underline{\vec{A}} = a + \vec{A}$

The multiplication of any two quaternions $\underline{\vec{A}} = a + \vec{A}$ and $\underline{\vec{B}} = a + \vec{B}$ is given by
[10-12]

$$\underline{\vec{A}\vec{B}} = (a + \vec{A})(b + \vec{B}) = ab - \vec{A}.\vec{B} + a\vec{B} + b\vec{A} + \vec{A} \times \vec{B} \tag{25}$$

Taking a = b = 0, we get from equation (25)

$$\underline{\vec{A}\vec{B}} = -\vec{A}.\vec{B} + \vec{A} \times \vec{B} \tag{26}$$

This product is called quaternion product. We can also generate a type of most general Lorentz transformation using this quarterion product.

In this case the velocity $\vec{V}$ of S' with respect to S has also three components, V_x, V_y, and V_z as the most general Lorentz transformation. Let in this case $\vec{Z}$ and $\vec{Z}'$ be the space parts in the S and S' frames, respectively. In this case, equation (14) is

$$(t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (t + \vec{Z})(0 + \vec{V})]$$

Here, $(t' + \vec{Z}')$, $(t + \vec{Z})$ and $(0 + \vec{V})$ are three quaternions [9]

According to the product of quaternions [10-12] we can write

$$(t + \vec{Z})(0 + \vec{V}) = -\vec{Z}.\vec{V} + t\vec{V} + \vec{Z} \times \vec{V} \tag{27}$$

From (27) and (14) we get

$$(t' + \vec{Z}') = \gamma[(t + \vec{Z}) - (-\vec{Z}.\vec{V} + t\vec{V} + \vec{Z} \times \vec{V})]$$

$$or,\ (t' + \vec{Z}') = \gamma(t + \vec{Z}.\vec{V}) + \gamma(\vec{Z} - t\vec{V} - \vec{Z} \times \vec{V}) \tag{28}$$

Equating the scalar and vector part from the both sides of equation (28), we can write

$$\begin{aligned} t' &= \gamma(t + \vec{Z}.\vec{V}) \\ \vec{Z}' &= \gamma(\vec{Z} - t\vec{V} - \vec{Z} \times \vec{V}) \end{aligned} \tag{29}$$

Similarly, we can show that

$$\begin{aligned} t &= \gamma(t' - \vec{Z}'.\vec{V}) \\ \vec{Z} &= \gamma(\vec{Z}' + t'\vec{V} + \vec{Z}' \times \vec{V}) \end{aligned} \tag{30}$$

Equations (29) and (30) are the quaternion Lorentz transformation.

2 RCIPROCAL PROPERTY OF DIFFERENT LORENTZ TRANSFORMATIONS

2.1 Rciprocal property of Special Lorentz Transformation

The velocity addition formula for the Special Lorentz transformation can be written as

$$W = \frac{x'}{t'} = \frac{x - Vt}{t - Vx/c^2} \tag{31}$$

Dividing numerator and denominator of equation by t we get

$$W = \frac{U - V}{1 - UV/c^2} \tag{32}$$

$$or,\ W = \frac{U - V}{1 - UV}$$

in unit of c.

If we replace U by P where $U\,P = 1$ then W will be change to W' where

$$W' = \frac{P - V}{1 - PV}$$

Reciprocal Property demands that if $U\,P=1$ then

$$W\,W'=\left(\frac{U-V}{1-UV}\right)\left(\frac{P-V}{1-PV}\right)=1$$

Now,

$$\begin{aligned}W\,W'&=\left(\frac{U-V}{1-UV}\right)\left(\frac{P-V}{1-PV}\right)\\&=\frac{UP-UV-VP+V^2}{1-PV-UV+UPV^2}\\&=\frac{1-UV-VP+V^2}{1-PV-UV+V^2}\end{aligned}$$

So, $W\,W'=1$ (33)

Consequently, the Special Lorentz Tranformation satisfies the Reciprocal property.

2.2 Rciprocal property of Most General Lorentz Tranformation

From the transformation equations of addition of velocities of Most General Lorentz Tranformation [2] we have

$$\vec{W}=\frac{\vec{X}'}{t'}=\frac{\vec{X}+\vec{V}[\{\frac{(\vec{X}.\vec{V})}{V^2}\}(\gamma-1)-t\gamma]}{\gamma\{t-\frac{(\vec{V}.\,\vec{X})}{c^2}\}}$$

Dividing numerator and denominator by t we get

$$\vec{W}=\frac{\vec{U}+\vec{V}\left[{\left(\vec{U}.\vec{V}\right)}/{V^2}\,(\gamma-1)-\gamma\right]}{\gamma\left(1-\vec{V}.\vec{U}\right)} \qquad (34)$$

in unit of c.

Where $\gamma=\dfrac{1}{\sqrt{1-{v^2}/{c^2}}}$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\,\vec{P}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}' = \frac{\vec{P} + \vec{V}\left[\frac{\left(\vec{P}.\,\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{V}.\vec{P}\right)}$$

Reciprocal Property demands that if $\vec{U}.\,\vec{P} = 1$ then

$$\vec{W}.\,\vec{W}' = \frac{\vec{U} + \vec{V}\left[\frac{\left(\vec{U}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{V}.\vec{U}\right)} . \frac{\vec{P} + \vec{V}\left[\frac{\left(\vec{P}.\,\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{V}.\,\vec{P}\right)} = 1$$

Now,

$$\vec{W}.\,\vec{W}' = \frac{\vec{U} + \vec{V}\left[\frac{\left(\vec{U}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{V}.\vec{U}\right)} . \frac{\vec{P} + \vec{V}\left[\frac{\left(\vec{P}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \vec{V}.\vec{P}\right)}$$

$$= \frac{\vec{U}.\vec{P} + \left(\vec{U}.\vec{V} + \vec{V}.\,\vec{P}\right)\left[\frac{\left(\vec{U}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right] + \left[\left(\vec{U}.\vec{V}\right)(\gamma - 1) - \gamma V^2\right]\left[\frac{\left(\vec{P}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma^2\left(1 - \vec{V}.\vec{U}\right)\left(1 - \vec{V}.\vec{P}\right)}$$

$$= \frac{1 + \left(\vec{U}.\vec{V} + \vec{V}.\,\vec{P}\right)\left[\frac{\left(\vec{U}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right] + \left[\left(\vec{U}.\vec{V}\right)(\gamma - 1) - \gamma V^2\right]\left[\frac{\left(\vec{P}.\vec{V}\right)}{V^2}(\gamma - 1) - \gamma\right]}{\gamma^2\left(1 - \left(\vec{V}.\vec{U}\right) + \left(\vec{V}.\vec{P}\right) + \left(\vec{V}.\vec{U}\right)\left(\vec{V}.\vec{P}\right)\right)}$$

So, $\vec{W}.\vec{W}' \neq 1$ (35)

Consequently, the Most General Lorentz Tranformation does not satisfy the Reciprocal property [13].

2.3 Rciprocal property of Mixed Number Lorentz Transformation

From the transformation equations of addition of velocities of mixed number Lorentz Tranformation [3] we have

$$\vec{W} = \frac{\vec{Z}'}{t'} = \frac{\gamma(\vec{Z} - t\vec{V} - i\vec{Z} \times \vec{V})}{\gamma(t - \vec{Z}.\vec{V})} = \frac{(\vec{Z} - t\vec{V} - i\vec{Z} \times \vec{V})}{(t - \vec{Z}.\vec{V})} \quad \text{in unit of c.} \qquad (36)$$

Dividing numerator and denominator of equation (36) by t we get

$$\vec{W}=\frac{\left(\vec{Z}/t-t\vec{V}/t-\left(i\vec{Z}\times\vec{V}\right)/t\right)}{\left(t/t-\vec{Z}.\vec{V}/t\right)}=\frac{\vec{U}-\vec{V}-i\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}} \qquad (37)$$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\vec{P}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}'=\frac{\vec{P}-\vec{V}-i\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}$$

Reciprocal Property demands that if $\vec{U}.\vec{P}=1$ then

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-i\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-i\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}=1$$

Now,

$$\vec{W}.\vec{W}'=\frac{\vec{U}-\vec{V}-i\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}}.\frac{\vec{P}-\vec{V}-i\vec{P}\times\vec{V}}{1-\vec{P}\cdot\vec{V}}$$

$$=\frac{\left(\vec{U}-\vec{V}-i\vec{U}\times\vec{V}\right).\left(\vec{P}-\vec{V}-i\vec{P}\times\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

$$=\frac{\left(\vec{U}\right).\left(\vec{P}-\vec{V}-i\vec{P}\times\vec{V}\right)-\left(\vec{V}\right).\left(\vec{P}-\vec{V}-i\vec{P}\times\vec{V}\right)-i\left(\vec{U}\times\vec{V}\right).\left(\vec{P}-\vec{V}-i\vec{P}\times\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

[Since $\vec{U}.\vec{P}=1$, $\left(\vec{U}\times\vec{V}\right).\vec{V}=0$, $\left(\vec{U}\times\vec{V}\right).\vec{P}=0$, $\vec{V}.\left(\vec{P}\times\vec{V}\right)=\vec{U}.\left(\vec{P}\times\vec{V}\right)=0$]

$$=\frac{1-\vec{U}\cdot\vec{V}-\vec{V}.\vec{P}+V^2-\left(\vec{U}\times\vec{V}\right).\left(\vec{P}\times\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

[Since $\vec{A}.\left(\vec{B}\times\vec{C}\right)=\vec{C}.\left(\vec{A}\times\vec{B}\right)$]

$$=\frac{1-\vec{U}\cdot\vec{V}-\vec{V}.\vec{P}+V^2-\vec{V}.\left\{\left(\vec{U}\times\vec{V}\right)\times\vec{P}\right\}}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

[Since $\left(\vec{A}\times\vec{B}\right)\times\vec{C}=\vec{B}\left(\vec{A}.\vec{C}\right)-\vec{A}\left(\vec{B}.\vec{C}\right)$]

$$=\frac{1-\vec{U}\cdot\vec{V}-\vec{V}.\vec{P}+V^2-\vec{V}.\left\{\vec{V}\left(\vec{U}\cdot\vec{P}\right)-\vec{U}\left(\vec{V}.\vec{P}\right)\right\}}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

$$=\frac{1-\vec{U}\cdot\vec{V}-\vec{V}.\vec{P}+V^2-V^2+\left(\vec{U}.\vec{V}\right)\left(\vec{V}.\vec{P}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

[since $\vec{U}.\vec{P}=1$ and $\vec{P}.\vec{V}=\vec{V}.\vec{P}$]

$$=\frac{1-\vec{U}\cdot\vec{V}-\vec{P}.\vec{V}+\left(\vec{U}.\vec{V}\right)\left(\vec{P}.\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}$$

So,

$$\vec{W}.\vec{W}'=\frac{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}{\left(1-\vec{U}\cdot\vec{V}\right)\left(1-\vec{P}\cdot\vec{V}\right)}=1 \tag{38}$$

Similarly

If we replace $\vec{V}$ by $\vec{Q}$ where $\vec{V}.\vec{Q}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$\vec{W}.\vec{W}'=1$

Consequently, the Mixed Number Lorentz Transformation satisfies the reciprocal property [13].

2.4 Rciprocal property of Geometric Product Lorentz Transformation

From the transformation equations of addition of velocities of Geometric product Lorentz Transformation [3] we have,

$$\vec{W}=\frac{\vec{Z}'}{t'}=\frac{\gamma(\vec{Z}-t\vec{V}-\vec{Z}\times\vec{V})}{\gamma(t-\vec{Z}.\vec{V})} \text{ in unit of c.} \tag{39}$$

$$=\frac{(\vec{Z}-t\vec{V}-\vec{Z}\times\vec{V})}{(t-\vec{Z}.\vec{V})}$$

Dividing numerator and denominator of equation (39) by t we get

$$\vec{W}=\frac{(\vec{Z}/t-t\vec{V}/t-\left(\vec{Z}\times\vec{V}\right)/t)}{(t/t-\vec{Z}.\vec{V}/t)}=\frac{\vec{U}-\vec{V}-\vec{U}\times\vec{V}}{1-\vec{U}\cdot\vec{V}} \tag{40}$$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\vec{P}=1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}' = \frac{\vec{P} - \vec{V} - \vec{P}\times\vec{V}}{1 - \vec{P}\cdot\vec{V}}$$

Reciprocal Property demands that if $\vec{U}.\,\vec{P} = 1$ then

$$\vec{W}.\,\vec{W}' = \frac{\vec{U} - \vec{V} - \vec{U}\times\vec{V}}{1 - \vec{U}\cdot\vec{V}}.\frac{\vec{P} - \vec{V} - \vec{P}\times\vec{V}}{1 - \vec{P}\cdot\vec{V}} = 1$$

Now,

$$\vec{W}.\,\vec{W}' = \frac{\vec{U} - \vec{V} - \vec{U}\times\vec{V}}{1 - \vec{U}\cdot\vec{V}}.\frac{\vec{P} - \vec{V} - \vec{P}\times\vec{V}}{1 - \vec{P}\cdot\vec{V}}$$

$$= \frac{\left(\vec{U} - \vec{V} - \vec{U}\times\vec{V}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right)}{\left(1 - \vec{U}\cdot\vec{V}\right)\left(1 - \vec{P}\cdot\vec{V}\right)}$$

$$= \frac{\left(\vec{U}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right) - \left(\vec{V}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right) - \left(\vec{U}\times\vec{V}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right)}{\left(1 - \vec{U}\cdot\vec{V}\right)\left(1 - \vec{P}\cdot\vec{V}\right)}$$

[Since $\vec{U}.\,\vec{P} = 1$, $\left(\vec{U}\times\vec{V}\right).\vec{V} = 0$, $\left(\vec{U}\times\vec{V}\right).\vec{P} = 0$, $\vec{V}.\left(\vec{P}\times\vec{V}\right) = \vec{U}.\left(\vec{P}\times\vec{V}\right) = 0$]

$$= \frac{1 - \vec{U}\cdot\vec{V} - \vec{V}.\,\vec{P} + V^2 + \left(\vec{U}\times\vec{V}\right).\left(\vec{P}\times\vec{V}\right)}{\left(1 - \vec{U}\cdot\vec{V}\right)\left(1 - \vec{P}\cdot\vec{V}\right)}$$

[Since $\vec{A}.\left(\vec{B}\times\vec{C}\right) = \vec{C}.\left(\vec{A}\times\vec{B}\right)$]

$$= \frac{1 - \vec{U}\cdot\vec{V} - \vec{V}.\,\vec{P} + V^2 + \vec{V}.\left\{\left(\vec{U}\times\vec{V}\right)\times\vec{P}\right\}}{\left(1 - \vec{U}\cdot\vec{V}\right)\left(1 - \vec{P}\cdot\vec{V}\right)}$$

[Since $\left(\vec{A}\times\vec{B}\right)\times\vec{C} = \vec{B}\left(\vec{A}.\vec{C}\right) - \vec{A}\left(\vec{B}.\vec{C}\right)$]

$$= \frac{1 - \vec{U}\cdot\vec{V} - \vec{V}.\,\vec{P} + V^2 + \vec{V}.\left\{\vec{V}\left(\vec{U}\cdot\vec{P}\right) - \vec{U}\left(\vec{V}.\,\vec{P}\right)\right\}}{\left(1 - \vec{U}\cdot\vec{V}\right)\left(1 - \vec{P}\cdot\vec{V}\right)}$$

$$= \frac{1 - \vec{U}\cdot\vec{V} - \vec{V}.\,\vec{P} + V^2 + V^2 - \left(\vec{U}.\vec{V}\right)\left(\vec{V}.\,\vec{P}\right)}{\left(1 - \vec{U}\cdot\vec{V}\right)\left(1 - \vec{P}\cdot\vec{V}\right)}$$

[since $\vec{U}.\,\vec{P} = 1$ and $\vec{P}.\,\vec{V} = \vec{V}.\,\vec{P}$]

$$= \frac{1 - \vec{U}\cdot\vec{V} - \vec{P}.\vec{V} + 2V^2 - \left(\vec{U}.\vec{V}\right)\left(\vec{P}.\vec{V}\right)}{\left(1 - \vec{U}\cdot\vec{V}\right)\left(1 - \vec{P}\cdot\vec{V}\right)}$$

So,

$$\vec{W}.\ \vec{W}' \neq 1 \tag{41}$$

Consequently, the Geometric product Lorentz Transformation does not satisfy the reciprocal property [13].

2.5 Rciprocal property of Quaternion Lorentz Transformation

From the transformation equations of addition of velocities of Quaternion Lorentz Transformation [3] we have

$$\begin{aligned}\vec{W} = \frac{\vec{Z}'}{t'} &= \frac{\gamma(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V})}{\gamma(t + \vec{Z}.\vec{V})} \quad \text{in unit of c.} \\ &= \frac{(\vec{Z} - t\vec{V} - \vec{Z}\times\vec{V})}{(t + \vec{Z}.\vec{V})}\end{aligned} \tag{42}$$

Dividing numerator and denominator of equation (42) by t we get

$$\vec{W} = \frac{\left(\vec{Z}/t - t\vec{V}/t - (\vec{Z}\times\vec{V})/t\right)}{\left(t/t + \vec{Z}.\vec{V}/t\right)} = \frac{\vec{U} - \vec{V} - \vec{U}\times\vec{V}}{1 + \vec{U}\cdot\vec{V}} \tag{43}$$

If we replace $\vec{U}$ by $\vec{P}$ where $\vec{U}.\ \vec{P} = 1$ then $\vec{W}$ will be change to $\vec{W}'$ where

$$\vec{W}' = \frac{\vec{P} - \vec{V} - \vec{P}\times\vec{V}}{1 + \vec{P}\cdot\vec{V}}$$

Reciprocal Property demands that if $\vec{U}.\ \vec{P} = 1$ then

$$\vec{W}.\vec{W}' = \frac{\vec{U} - \vec{V} - \vec{U}\times\vec{V}}{1 + \vec{U}\cdot\vec{V}} . \frac{\vec{P} - \vec{V} - \vec{P}\times\vec{V}}{1 + \vec{P}\cdot\vec{V}} = 1$$

Now,

$$\begin{aligned}\vec{W}.\ \vec{W}' &= \frac{\vec{U} - \vec{V} - \vec{U}\times\vec{V}}{1 + \vec{U}\cdot\vec{V}} . \frac{\vec{P} - \vec{V} - \vec{P}\times\vec{V}}{1 + \vec{P}\cdot\vec{V}} \\ &= \frac{\left(\vec{U} - \vec{V} - \vec{U}\times\vec{V}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right)}{\left(1 + \vec{U}\cdot\vec{V}\right)\left(1 + \vec{P}\cdot\vec{V}\right)} \\ &= \frac{\left(\vec{U}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right) - \left(\vec{V}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right) - \left(\vec{U}\times\vec{V}\right).\left(\vec{P} - \vec{V} - \vec{P}\times\vec{V}\right)}{\left(1 + \vec{U}\cdot\vec{V}\right)\left(1 + \vec{P}\cdot\vec{V}\right)}\end{aligned}$$

[Since $\vec{U}.\ \vec{P} = 1\ , \left(\vec{U} \times \vec{V}\right).\vec{V} = 0\ \ , \left(\vec{U} \times \vec{V}\right).\vec{P} = 0\ , \vec{V}.\left(\vec{P} \times \vec{V}\right) = \vec{U}.\left(\vec{P} \times \vec{V}\right) = 0$]

$$= \frac{1 - \vec{U} \cdot \vec{V} - \vec{V}.\ \vec{P} + V^2 + \left(\vec{U} \times \vec{V}\right).\left(\vec{P} \times \vec{V}\right)}{\left(1 + \vec{U} \cdot \vec{V}\right)\left(1 + \vec{P} \cdot \vec{V}\right)}$$

[Since $\vec{A}\ .\ \left(\vec{B} \times \vec{C}\right) = \vec{C}.\ \left(\vec{A} \times \vec{B}\right)$]

$$= \frac{1 - \vec{U} \cdot \vec{V} - \vec{V}.\ \vec{P} + V^2 + \vec{V}.\ \left\{\left(\vec{U} \times \vec{V}\right) \times \vec{P}\right\}}{\left(1 + \vec{U} \cdot \vec{V}\right)\left(1 + \vec{P} \cdot \vec{V}\right)}$$

[Since $\left(\vec{A} \times \vec{B}\right) \times \vec{C} = \vec{B}\left(\vec{A}.\vec{C}\right) - \vec{A}\left(\vec{B}.\vec{C}\right)$]

$$= \frac{1 - \vec{U} \cdot \vec{V} - \vec{V}.\ \vec{P} + V^2 + \vec{V}.\ \left\{\vec{V}\left(\vec{U} \cdot \vec{P}\right) - \vec{U}\left(\vec{V}.\ \vec{P}\right)\right\}}{\left(1 + \vec{U} \cdot \vec{V}\right)\left(1 + \vec{P} \cdot \vec{V}\right)}$$

$$= \frac{1 - \vec{U} \cdot \vec{V} - \vec{V}.\ \vec{P} + V^2 + V^2 - \left(\vec{U}.\vec{V}\right)\left(\vec{V}.\vec{P}\right)}{\left(1 + \vec{U} \cdot \vec{V}\right)\left(1 + \vec{P} \cdot \vec{V}\right)}$$

[since $\vec{U}.\ \vec{P} = 1$ and $\ \vec{P}.\vec{V} = \vec{V}\ .\ \vec{P}$]

$$= \frac{1 - \vec{U} \cdot \vec{V} - \ \vec{P}\ .\vec{V} + 2V^2 - \left(\vec{U}.\vec{V}\right)\left(\ \vec{P}.\vec{V}\right)}{\left(1 + \vec{U} \cdot \vec{V}\right)\left(1 + \vec{P} \cdot \vec{V}\right)}$$

So, $\vec{W}.\ \vec{W}' \neq 1$ (44)

Consequently, the Quaternion Lorentz Transformation does not satisfy the reciprocal property [13].

3 Conclusion

The reciprocal property of different Lorentz transformations has been discussed and we have obtained that

1. Special and Mixed Number Lorentz Transformations satisfy the Reciprocal property.
2. Most General, Geometric product and Quaternion Lorentz Transformations do not satisfy the reciprocal property.

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